The Laplace-Runge-Lenz Vector

2022 Summer Camp

$$\vec{\varepsilon} = \frac{\vec{L} \times \vec{v}}{GMm} + \vec{e}_r = \text{const.}$$

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Introduction

Background

- The LRL vector is a conservation law in orbital mechanics.
- It is most related to the eccentricity of the orbit and is given by the equation
$$\vec{\varepsilon} = \frac{\vec{L} \times \vec{v}}{GMm} + \vec{e}_r = \text{const.}$$
- Here, $\vec{L}$ is the angular momentum of the object, $\vec{v}$ is the velocity, G is the universal gravitational constant, $\vec{e}_r$ represents the radial vector, and M and m are the big and little masses respectively.

Eccentricity

- Eccentricity is a fundamental parameter used to describe the shape of an orbit. In the context of celestial mechanics, eccentricity refers to how elongated or circular an elliptical orbit is. It is a dimensionless quantity that ranges from 0 to 1, where:
  - An eccentricity of 0 represents a perfectly circular orbit.
  - An eccentricity between 0 and 1 represents an elliptical orbit, with
  - higher values indicating a more elongated shape.
  - An eccentricity of 1 represents a parabolic orbit, which is the boundary between elliptical and hyperbolic orbits.
  - An eccentricity greater than 1 represents a hyperbolic orbit, where the object follows a path that is open and unbounded.

Eccentricity Continued

- The eccentricity of an orbit can be calculated using various methods, but one common approach is to use the semi-major axis (a) and semi-minor axis (b) of the ellipse.

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}.$$

- Eccentricity has significant implications for the dynamics and characteristics of celestial bodies in orbit. For example, orbits with higher eccentricities have more significant variations in distance between the orbiting object and the central body.

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- $\vec{v} \times \frac{d\vec{L}}{dt} = 0$

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- $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\left(\frac{d|\vec{r}|}{dt}\vec{e}_r + |\vec{r}|\frac{d\vec{r}}{dt}\right) = mr^2\left(\vec{e}_r + \frac{d\vec{e}_r}{dt}\right)$

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- $\vec{C} = \vec{v} \times \vec{L} - GMm\vec{e}_r$.

Additional Information

General Things

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- The LRL vector is directed in the same direction (left or right) throughout an orbit's path.
- LRL vector is conserved for all types of orbits (ellipse, parabola, hyperbola, etc).

Hodograph

- Take cross product of $\vec{A}$ with angular momentum $\vec{L}$

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- Since $\vec{L} \perp \vec{p}$, then $\vec{L} \cdot \vec{p} = 0$:

$$-lA\vec{e}_y = l^2 \vec{p} - GMm^2 \vec{e}_\theta.$$

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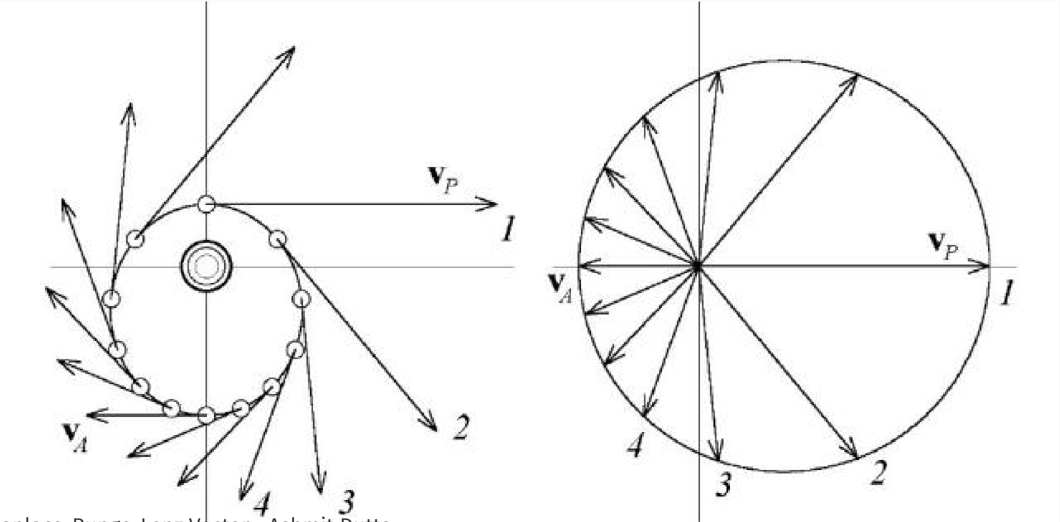
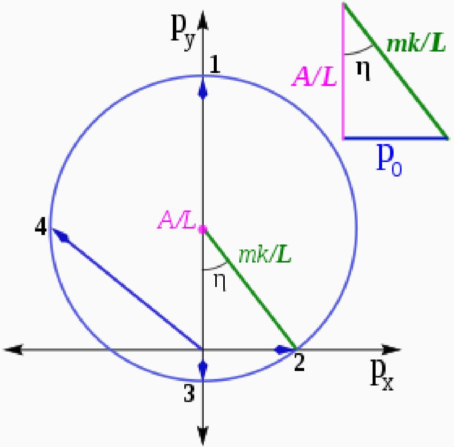
- $\left|\vec{p} + \frac{A}{l}\vec{e}_y\right| = \frac{GMm^2}{l} \implies p_x^2 + \left(p_y - \frac{A}{l}\right)^2 = \left(\frac{GMm^2}{l}\right)^2.$

Visualization



Visualization Cont.

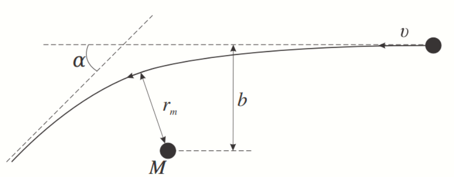


Problems

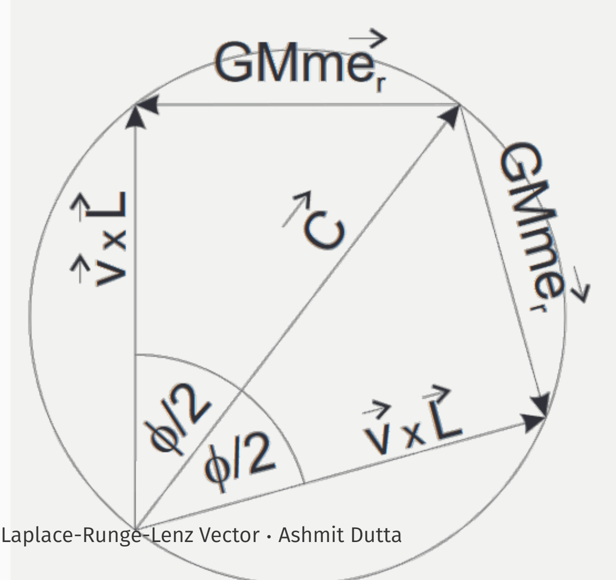
2017 APhO Problem 2

Let us study a simplified version of a galaxy. You can ignore the velocities of the stars in the galaxy. All the stars are of mass m. Consider a super massive blackhole (SBH) of mass M ($M \gg m$) moving with a velocity v through the galaxy. We are working the reference frame of SBH.

Consider the transit of one star with impact parameter b. Assume that $b \gg b_1 = GM/v^2$. The angular deflection of the star is $\alpha = kb_1/b$, where k is some coefficient. Find the value of k.



Creating a ”Phasor” Diagram



2021 Physics Cup Problem 2 (PoTW)

At two different points in its orbit, a comet has velocities $\vec{v}_1$ and $\vec{v}_2$. If:

- $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and;
- $|\vec{v}_1| = 2|\vec{v}_2|$, what is the smallest possible eccentricity of the orbit?

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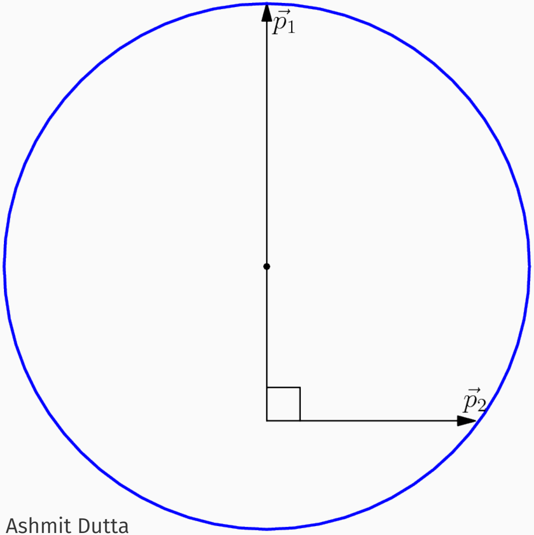

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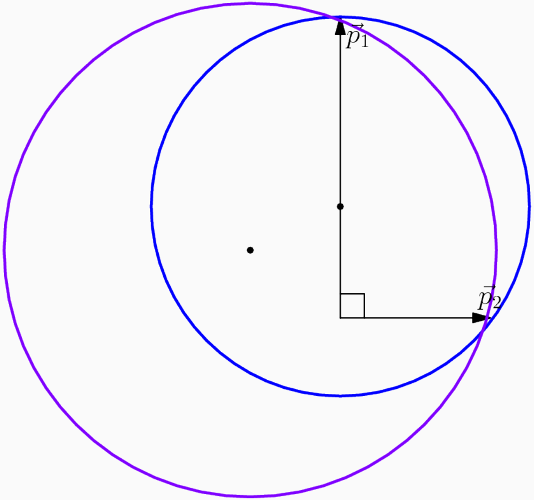

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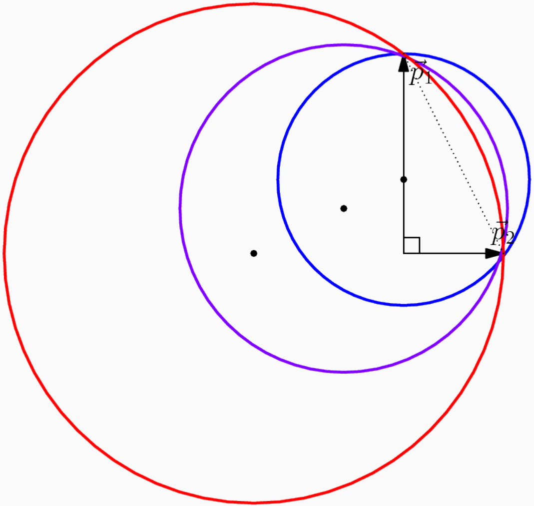

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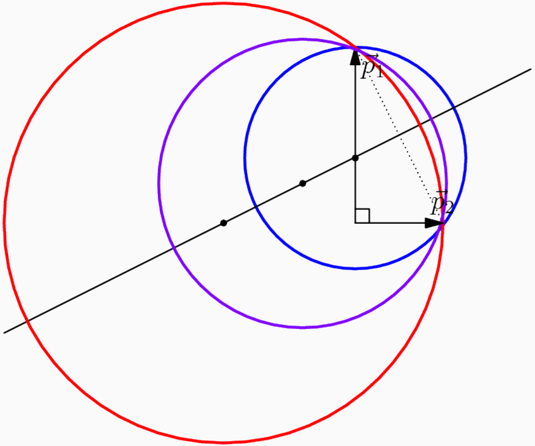

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- We then can find the equation for the line that adjoins all the centers by generalizing the center to be at a coordinate (x, y).
- Then we can write that

$$\sqrt{(p_2 - x)^2 + y^2} = \sqrt{(p_1 + y)^2 + x^2} \implies y = -\frac{p_2}{p_1}x + \frac{p_2^2 - p_1^2}{p_1}.$$

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- Now that we have found this line, note that the magnitude of the eccentricity vector is given to be $\varepsilon = A/GMm^2$.
- If we look at the geometry of a circular hodograph, it is apparent that $\varepsilon = \cos\varphi = y/R$ where $y = A/l$ or the displacement of the hodograph from the origin, R is the radius of the hodograph, and φ is the angle between both scalars. We know that $R = \sqrt{y^2 + (p_2 - x)^2}$.

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- Since $\varepsilon > 0$, this is the same as minimizing

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- This means that $\varepsilon = \sqrt{1/5} \approx 0.44$.